Dynamical Systems
Tutorial 11 - Lyapunov exponents and other indicators of chaos

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Introduction

Recall that we defined chaos using the definition from Devaney:

Let V be a set. $f : V \to V$ is said to be chaotic on V if:

1. f has sensitive dependence on initial conditions:

$$\exists\delta > 0 : \forall x \in V, N\ neighborhood\ of\ x, \exists y \in V : \|f^n(x) - f^n(y)\| > \delta$$

   We emphasize that not all points near x need eventually separate from x under iteration, but there must be at least one such point in every neighborhood of x. If a map possesses sensitive dependence on initial conditions, then for all practical purposes, the dynamics of the map defy numerical computation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may bear no resemblance whatsoever with the real orbit.

2. f is topologically transitive:

$$\forall open\ sets\ U, W \subset V, \exists k > 0 : f^k(U) \cap W \neq \emptyset$$

   Enough to show that the map has a dense orbit.

3. periodic points are dense in V

   (may be switched with condition of a compact phase space - this is the definition in Meiss. For a discussion of the different definitions, see "Aulbach, B., and Kieninger, B. (2001). On three definitions of chaos.")

The concept of "sensitive dependence" requires that nearby orbits eventually separate; however, the rate of separation is not specified. Also, this is not enough for chaos. There are non-chaotic systems whose trajectories separate exponentially in time (a linear hyperbolic flow $\dot{x} = Kx$, $K > 0$) and those that separate at a polynomial rate (like the flow on the cylinder: $\dot{z} = 0, \dot{\theta} = z$).

Given a dynamical system, how do we know if it's chaotic? We would like to have a heuristic measure - this is the Lyapunov exponent. Positive values signify exponential separation of close initial conditions. Generally, there appears to be a dichotomy between systems for which nearby orbits separate linearly and truly chaotic systems whose orbits separate exponentially. We say that two orbits of a flow separate exponentially when $|\phi_t(y) - \phi_t(x)| \sim ce^{\lambda t}$ and $\lambda > 0$. In a compact domain, this separation rate cannot go on forever. Thus, exponential separation is required for initially infinitesimally close trajectories in chaotic systems. However, we emphasize that a positive value of λ do not guarantee chaos - thus the Lyapunov exponent is only an indicator of chaos.

Lyapunov exponents

Lyapunov exponent in a map (Dorfman)

Consider a differentiable map $M : (0,1) \to (0,1)$. Examine a small interval $(x_0, x_0 + \delta x_0)$. M maps this interval to $(M(x_0), M(x_0 + \delta x_0)) = (x_1, x_1 + \delta x_1)$, then to $(M(x_1), M(x_1 + \delta x_1)) = (x_2, x_2 + \delta x_2)$, etc. After a large number of steps n, the interval of length δx_0 may become exponentially large or small, $\delta x_0 \to \delta x_n \approx \delta x_0 \exp(n\lambda(x_0))$, with $\lambda(x_0)$ larger or smaller than zero. If, after a number of steps, the interval gets folded or cut, then the scaling factor may no longer be approximated in this simple way. Therefore the limit $\delta x_0 \to 0$ is taken to obtain an expression for $\lambda(x_0)$.

Thus, we define a Lyapunov exponent $\lambda(x_0)$ at a point x_0 by:

$$\lambda(x_0) = \limsup_{n\to\infty} \lim_{\delta x_0 \to 0} \frac{1}{n} \ln \left| \frac{M^n(x_0 + \delta x_0) - M^n(x_0)}{\delta x_0} \right|, \quad (1)$$

where $M^n(x) = M(M(...(M(x)...)))$, that is, the nth iterate of the map at point x.

The Lyapunov exponent can be written more simply as a derivative,

$$\lambda(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \left| \frac{dM^n(x_0)}{dx_0} \right|. \quad (2)$$

Using the chain rule, $\frac{dM^n(x_0)}{dx_0} = M'(x_{n-1})M'(x_{n-2})...M'(x_0)$, so a convenient form for the Lyapunov exponent of this map is

$$\lambda(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} M'(x_i) \right| = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |M'(x_i)|. \quad (3)$$

Thus, the Lyapunov exponent measures the rate of separation or approach of two nearby phase points as they each follow the trajectory determined by their respective initial conditions. A positive Lyapunov exponent means that two nearby points will separate exponentially with the number of steps, or, for a flow, with time. This is an expression of the sensitivity of the dynamics to the initial conditions - a small range of initial states will sample a large region of phase space. Extending this idea to a multidimensional map is similar.

Example: Tripling map on a torus

For the map $M : \theta \to 3\theta \mod (2\pi)$, it's easy to see that $\lambda(\theta) = \ln 3$ for all values of θ.

Example: Baker's map

There are 2 Lyapunov exponents, $\pm \ln 2$.

Lyapunov exponents in a multidimensional flow

In order to obtain an asymptotic expression for λ in the case of a general flow $\dot{x} = f(x)$, we want to linearize around an arbitrary orbit. Given an orbit $\varphi_t(x^*)$, consider a trajectory that starts close, $\varphi_t(x^* + \varepsilon v_0)$. For a C^1 flow,

$$\varphi_t(x^* + v_0) = \varphi_t(x^*) + D_x\varphi_t(x^*)v_0 + o(|v_0|). \tag{4}$$

The deviation vector, defined as the first order approximation of the difference between the trajectories $\varphi_t(x^* + v_0) - \varphi_t(x^*)$, satisfies

$$v(t) = D_x\varphi_t(x^*)v_0 \tag{5}$$

and its time evolution is

$$\dot{v} = Df(\varphi_t(x^*))v \equiv A(t)v. \tag{6}$$

For simplicity of our notation, we define the matrix $\Phi(t,x) = D_x\varphi_t(x)$. By this definition, $v(t) = \Phi(t,x)v_0$, and $\dot{\Phi} = A\Phi$.

In the case of linearization around a fixed point, A is independent of time and its eigenvalues can be found, which describe the linearized motion around the fixed point. However, in the general case we show here, A depends on time.

Now, we want to define a Lyapunov exponent as a number λ such that, asymptotically, $|\Phi(t,x)v| \sim e^{\lambda t}|v|$. Why should such a λ exist? Could the separation be faster than exponential?

Lemma 7.3 (M): If $\Phi(t,x)$ solves $\dot{\Phi} = A\Phi$, and $\|A\| \leq K \forall t$, then for all v exist constants c, c' such that

$$C'e^{-Kt} \leq |\Phi(t,x)v| \leq Ce^{Kt} \; \forall t \geq 0. \tag{7}$$

The condition of boundedness of $\|A\|$ is satisfies for a compact set and a continuous f of the flow. We won't prove this lemma, the proof appears in (M). However we use it to show that $\frac{\ln|\Phi v|}{t}$ is bounded for $t \geq 0$, for any v.

Thus, we define the Lyapunov exponents as the supremum limits:

$$\lambda(x,v) = \limsup_{t\to\infty} \frac{1}{t}\ln|\Phi(t,x)v| = \limsup_{t\to\infty, v\to 0} \frac{1}{t}\ln\left|\frac{\varphi_t(x+v) - \varphi_t(x)}{v}\right|. \tag{8}$$

The lemma guarantees that this limit exists.

Note that the value of λ depends on the direction of v! In fact, for an n-dimensional phase space, a system has in general n distinct Lyapunov exponents relating to the different linearly independent directions for the deviation vector:

Lemma 7.5 (M): If $\varphi_t(x)$ is bounded then it has at most n distinct Lyapunov exponents.

Proof: Since the flow is bounded, the Jacobian is bounded and the supremum limit exists for all v. Taking two linearly independent vectors v_1 and v_2 with respective Lyapunov exponents λ_1 and λ_2, their linear combination $v = av_1 + bv_2$ will grow asymptotically as $\max\{\lambda_1, \lambda_2\}$. Since there are n linearly independent vectors, there are at most n Lyapunov exponents. □

Example: The simple linear one-dimensional ODE

$$\dot{v} = (\cos(\ln|t|) + \sin(\ln|t|))\, v$$

has the general solution $v(t) = \exp(t \sin(\ln|t|))\, v_o$ (ignoring the fact that the vector field is not defined at $t = 0$). For this system the fundamental matrix is simply the scalar $\exp(t \sin(\ln|t|))$, and the Lyapunov spectrum is

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\exp(t \sin \ln|t|) x_0) = 1 \tag{9}$$

Comments:

* The largest Lyapunov exponent is known as the Lyapunov Characteristic Exponent, or LCE. A positive LCE indicates chaos.

* For any flow generated by the equation $\dot{x} = f(x)$, at least one of the Lyapunov exponents must vanish, because in the direction of the flow, the deviation vector grows linearly with time.

* For a volume-preserving flow (such as a Hamiltonian flow), the sum of the Lyapunov exponents must be zero. For a dissipative system, the sum is negative.

* In Hamiltonian systems with n degrees of freedom there are $2n$ Lyapunov exponents, and they come in pairs: if we arrange them in increasing order, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, then $\lambda_m = -\lambda_{2n-m}$, and there are at least two directions in which the exponent disappears - in the direction of the flow and in the direction of the energy gradient. This is an expression of the Liouville theorem.

Numerical calculation of the Lyapunov Characteristic Exponent

For almost all initial tangent vectors v, $\lambda(x, v)$ equals the LCE, because almost all initial tangent vectors will have some component in the maximal direction. Therefore, given a dynamical system, choose some v_0 around a (numerically) calculated orbit $x(t)$, and integrate the equation for $v(t)$, $\dot{v} = Mv$. Thus obtain numerically the deviation distance, $d(t) = |v(t)|$, where $d_0 = 1$ for convenience.

Potential problem: If the norm $d(t)$ increases exponentially, there will be a risk of computational errors. Instead, the scheme suggested by Benettin et al. 1976 is commonly used. Choose a small fixed interval τ, and renormalize v to unity every τ time units. Thus, iteratively compute $d_k = |v_{k-1}(\tau)|$ where $v_k(0) = \frac{v_{k-1}(\tau)}{d_k}$, and define the averaged Lyapunov exponent

$$\lambda_n = \frac{1}{n\tau} \sum_{i=1}^{n} \ln d_i. \tag{10}$$

If τ is small enough, it can be shown that

$$\lambda_\infty = \lim_{n\to\infty} \lambda_n = \lambda_1 \tag{11}$$

exists and is independent of τ.

There are many alternative methods to calculate the largest Lyapunov exponent and the entire spectrum; see review by Skokos for a partial summary.

Kolmogorov-Sinai entropy

Heuristic considerations:

When a system exhibits exponential separation, one gains information about initial conditions by iterating the system. That is, suppose we can distinguish two points only if they are separated by a distance δ, the resolution parameter, and suppose we are given an initial set of this order of magnitude, denoted A. Then we cannot resolve two points in A then, but after a time t the initial set will be stretched to a length of order $\delta \exp(\lambda_+ t)$, and we can easily resolve the images of points in the initial set. Thus, by looking at successive images of the initial set we learn more and more about the location of points in the initial region, and the information is growing at an exponential rate. This rate is measured by the Kolmogorov-Sinai (KS) entropy, denoted h_{KS}.

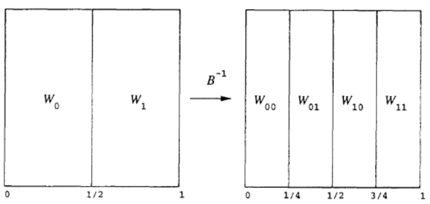
Definition of the KS entropy, described by example of the Baker's Map (Dorfman)

Consider a phase space Λ of finite total measure. Suppose we decompose Λ into a collection of non-overlapping sets $\{W_i\}$ s.t.

$$\{W_i : \Lambda = \cup_i W_i, W_i \cap W_j = \emptyset for\, i \neq j, \mu(W_i) > 0\}.$$

This is a partition of Λ. Given a partition, we can create finer and finer partitions by examining the pre-images of a partition and taking intersections of the original partition with its pre-images.

As an example, consider the partition of the unit square of the baker's map W_0, W_1:



The inverse Baker's Map maps the two sets to $B^{-1}W_i$. The intersections of W_i with $B^{-1}W_j$ leads to a new partition of the unit square into 4 sets:

$$W_{ij} = \{x : x \in W_i, B(x) \in W_j\}/$$

This partition is the V-sum of the intersections of the partitions. By running this procedure backwards and taking further intersections, finer partitions are obtained, and we get a collection of partitions that contain more and more details about the trajectory of a point:

$$\{\{W_i\}.\{W_i \cap B^{-1}(W_j)\}, \{W_i \cap B^{-1}(W_j) \cap B^{-2}(W_k)\}, \ldots\}$$

If we know $x_0 \in W_0 \cap B^{-1}(W_1) \cap B^{-2}(W_0)$, then we know that $B(x_0) \in W_1$ and $B^2(x_0) \in W_0$. Thus, by identifying the element of the partition to which a point belongs, one can map out the entire history of a point.

What we would like to know (and quantify) is how fine the partition is becoming - indicator of mixing of the phase space and of our ability to use larger and larger parts of the trajectory to uniquely specify a particular initial point.

KS defined the entropy of a partition as

$$H(\{W_i\}) = -\sum_i \mu(W_i \ln(\mu(W_i)).$$

When the partition is the trivial partition, $W = \Lambda$, then $H(\Lambda) = 0$. For the example of the Baker's Map, the partitions have entropies:

$$H_1 \equiv H(W) = -\frac{1}{2}(\ln(\frac{1}{2}) + \ln(\frac{1}{2}))$$

$$H_2 \equiv H(W \cap B^{-1}(W)) = -4\frac{1}{4}\ln(\frac{1}{4})$$

etc. To indicate how much information is gained per step, define

$$h = \lim_{n\to\infty} \frac{1}{n} H_n.$$

This is, in a sense, the measure for the rate at which information is produced for someone who observes the system with a limited resolution.

The KS entropy is defines as the supremum of the above expression over all possible (initial) partitions:

$$h_{KS} = \sup_W h.$$

A partition that gives the KS entropy directly is called a generating partition.

In the rotation map, $x_n \to x_n + \alpha \mod 1$, a partition into two parts will stay a partition into two parts when the system is run backwards. After n iterations, the circle will be partitioned into $2n$ intervals, the entropy depends logarithmically on n and the KS entropy is 0.

In the Baker's Map, the partition we showed is the generating partition, and leads directly to $h_{KS} = \ln 2$. Note that this is exactly the value of the positive Lyapunov exponent of the same system - this is not a coincidence.

Pesin's theorem

For closed Anosov hyperbolic systems, the KS entropy is equal to the sum of the positive Lyapnov exponents.

* A system with a positive KS entropy is called a K-system.